

On a symmetrization of hemiimplicative semilattices

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Abstract

A hemiimplicative semilattice is a bounded semilattice $(A, \wedge, 1)$ endowed with a binary operation \rightarrow , satisfying that for every $a, b, c \in A$, $a \leq b \rightarrow c$ implies $a \wedge b \leq c$ (that is to say, one of the conditionals satisfied by the residuum of the infimum) and the equation $a \rightarrow a = 1$. The class of hemiimplicative semilattices form a variety. These structures provide a general framework for the study of different structures of interest in algebraic logic. In any hemiimplicative semilattice it is possible to define a derived operation by $a \sim b := (a \rightarrow b) \wedge (b \rightarrow a)$. Endowing $(A, \wedge, 1)$ with the binary operation \sim results again a hemiimplicative semilattice, which also satisfies the identity $a \sim b = b \sim a$. We call the elements of the subvariety of hemiimplicative semilattices satisfying $a \rightarrow b = b \rightarrow a$, a symmetric hemiimplicative semilattice. In this article, we study the correspondence assigning the symmetric hemiimplicative semilattice $(A, \wedge, \sim, 1)$ to the hemiimplicative semilattice $(A, \wedge, \rightarrow, 1)$. In particular, we characterize the image of this correspondence. We also provide many new examples of hemiimplicative semilattice structures on any bounded semilattice (possibly with bottom). Finally, we characterize congruences on the classes of hemiimplicative semilattices introduced as examples and we describe the principal congruences of hemiimplicative semilattices.

1 Introduction

Recall that a structure (A, \leq, \cdot, e) is said to be a partially ordered monoid if (A, \leq) is a poset, (A, \cdot, e) is a monoid and for all $a, b, c \in A$, if $a \leq b$ then $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$. Although commutativity do not play any special feature in the discussion that follow, we shall assume in this article that all monoids are commutative in order to make the exposition more clear.

Let us also recall that the residuum (when it exists) of the monoid operation of a partially ordered monoid (A, \leq, \cdot, e) is a binary operation \rightarrow on A such that for every a, b and c in A ,

$$a \cdot b \leq c \text{ if and only if } a \leq b \rightarrow c.$$

Note that previous equivalence can be seen as the conjunction of the following conditionals:

- (r) If $a \cdot b \leq c$ then $a \leq b \rightarrow c$ and
- (l) If $a \leq b \rightarrow c$ then $a \cdot b \leq c$.

This suggest us to consider binary operations \rightarrow satisfying either (r) or (l) above. We call such operations r-hemiresidua (l-hemiresidua) of the monoid operation respectively.

Let $\mathbf{A} = (A, \leq, \cdot, e)$ be a partially ordered monoid. An *r-hemiresiduated monoid* is the expansion of \mathbf{A} with an r-hemiresiduum. Similarly, we define an *l-hemiresiduated monoid*. Clearly, every partially ordered residuated monoid is both an r-hemiresiduated monoid and an l-hemiresiduated monoid. Straightforward calculations show that the *BCK*-algebras with meet [6, 9] are examples of r-hemiresiduated monoids which in general are not residuated. Some examples of l-hemiresiduated monoids which in general are not residuated can be found for instance in [13].

Remark 1.1. Let (A, \leq, \cdot, e) be a partially ordered monoid and \rightarrow a binary operation on A . Then $(A, \leq, \cdot, \rightarrow, e)$ is an l-hemiresiduated monoid if and only if $a \cdot (a \rightarrow b) \leq b$ for every $a, b \in A$. In order to prove it, suppose that $(A, \leq, \cdot, \rightarrow, e)$ is an l-hemiresiduated monoid. For every $a, b \in A$ we have that $a \rightarrow b \leq a \rightarrow b$, so $a \cdot (a \rightarrow b) \leq b$. Conversely, suppose that $a \cdot (a \rightarrow b) \leq b$ for any $a, b \in A$. Let $a, b, c \in A$ such that $a \leq b \rightarrow c$. Then $a \cdot b \leq b \cdot (b \rightarrow c) \leq c$, so $a \cdot b \leq c$. Therefore, $(A, \leq, \cdot, \rightarrow, e)$ is an l-hemiresiduated monoid.

In this work we shall be particularly interested in the case that the partially ordered monoid is idempotent; more precisely, a meet semilattice whose order agrees with that induced by the infimum. We call *r-hemiresiduated semilattices* (*l-hemiresiduated semilattices*) to the expansion of these structures with a r-hemiresiduum (l-hemiresiduum). Throughout this paper we write semilattice in place of meet semilattice. A semilattice (A, \wedge) is said to be bounded if it has a greatest element, which will be denoted by 1; in this case we write $(A, \wedge, 1)$. A *l-hemiimplicative semilattice* is an l-hemiresiduated semilattice satisfying some additional conditions. Since we shall only consider l-hemi-implicative semilattice, in what follows we shall omit the l- prefix.

Definition 1.2. A *hemiimplicative semilattice* is an algebra $(A, \wedge, \rightarrow, 1)$ of type $(2, 2, 0)$ which satisfies the following conditions:

- (H1) $(A, \wedge, 1)$ is a bounded semilattice,
- (H2) for every $a \in A$, $a \rightarrow a = 1$ and
- (H3) for every $a, b, c \in A$, if $a \leq b \rightarrow c$ then $a \wedge b \leq c$.

Notice that (H3) is the condition (1) for the case in which $\cdot = \wedge$. Hemiimplicative semilattices were called weak implicative semilattices in [14]. Let (A, \wedge) be a semilattice and \rightarrow a binary operation. Then by Remark 1.1 we have that A satisfies (H3) if and only if for every $a, b \in A$ the equation $a \wedge (a \rightarrow b) \leq b$ is satisfied. In every hemiimplicative semilattice we define $a \leftrightarrow b := (a \rightarrow b) \wedge (b \rightarrow a)$. We write hIS for the variety of hemiimplicative semilattices.

Remark 1.3. Let $A \in \text{hIS}$ and $a, b \in A$. Then $a = b$ if and only if $a \leftrightarrow b = 1$. We also have that $1 \rightarrow a \leq a$.

Recall that an implicative semilattice [4, 10] is an algebra (A, \wedge, \rightarrow) of type $(2, 2)$ such that (A, \wedge) is semilattice, and for every $a, b, c \in H$ it holds that $a \wedge b \leq c$ if and only if $a \leq b \rightarrow c$. Every implicative semilattice has a greatest element. In this paper we shall include the greatest element in the language of the algebras. We write IS for the variety of implicative semilattices.

Remark 1.4. Clearly, IS is a subvariety of hIS. Other examples of hemiimplicative semilattices are the $\{\wedge, \rightarrow, 1\}$ -reduct of semi-Heyting algebras [11, 12] and the $\{\wedge, \rightarrow, 1\}$ -reduct of some algebras with implication [2], as for example the $\{\wedge, \rightarrow, 1\}$ -reduct of RWH-algebras [1].

In [7] Jenei shows that the class of BCK algebras with meet is term equivalent to the class of equivalential equality algebras, and he defines the equivalence operation \sim in terms of the implication in the usual way; i.e., $a \sim b := a \leftrightarrow b$. Some of these ideas were generalized and studied for pseudo BCK-algebras [3, 5, 8].

In particular, the variety of implicative semilattices is term equivalent to a subvariety of that of equivalential equality algebras. Let us write ES for this subvariety. The algebras in the class ES satisfy:

- a) $a \sim b = b \sim a$,
- b) $a \sim a = 1$,
- c) $a \wedge (a \sim b) \leq b$.

On the other hand, implicative semilattices satisfy b) and c) above, but of course not necessarily a). Hence there seems to be a common frame for both classes of algebras, where the algebras in ES may be seen as elements with a symmetric implication and the construction $a \sim b := a \leftrightarrow b$ a sort of symmetrization of the original implication. In this paper we explore a convenient framework where the aforementioned intuitions could be made precise. Most results concerning the relation between implicative semilattices and the class ES are part of the folklore. However, for the sake of completeness, we shall recall some basic results in Section 2.

In Section 3 some subvarieties of hemiimplicative semilattices are presented. New examples of hemiimplicative semilattices are provided, by defining a suitable structure on any bounded (sometimes with bottom) semilattices. The relationship between the variety of hemiimplicative semilattices and its subvariety of symmetric elements is studied.

In Section 4 we characterize congruences on the classes of hemiimplicative semilattices introduced in the examples of Section 3 and we describe the principal congruences of hemiimplicative semilattices.

2 Relation between IS and ES

As we have mentioned before, the relation between implicative semilattices and the class ES is part of the folklore. However, for the sake of completeness, we shall make explicit some details about this relation.

Remark 2.1. a) In every implicative semilattice $(A, \wedge, \rightarrow, 1)$ we have that $(A, \wedge, 1)$ is a bounded semilattice and $a \leftrightarrow a = 1$ for every $a \in A$. We also have that for every $a, b, c \in A$, $c \leq a \leftrightarrow b$ if and only if $a \wedge c = b \wedge c$.

b) Implicative semilattices satisfy $a \rightarrow b = a \leftrightarrow (a \wedge b)$ for every $a, b \in A$.

c) Consider an algebra $(A, \wedge, \sim, 1)$ of type $(2, 2, 0)$ such that (A, \wedge) is a semilattice. For every $a, b, c \in A$ we consider the following conditions: 1) $a \wedge (a \sim b) = b \wedge (a \sim b)$ and 2) if $a \wedge c = b \wedge c$ then $c \leq a \sim b$. For every $a, b, c \in A$ conditions 1) and 2) are satisfied if and only if we have that $a \sim b = \max \{c \in A : a \wedge c = b \wedge c\}$ for every $a, b \in A$.

In the following proposition we consider a particular class of algebras.

Proposition 2.2. *Let $(A, \wedge, \sim, 1)$ be an algebra of type $(2, 2, 0)$ such that satisfies the following conditions:*

1) $(A, \wedge, 1)$ is a bounded semilattice,

2) $a \sim a = 1$,

3) $a \wedge (a \sim b) = b \wedge (a \sim b)$,

4) if $a \wedge c = b \wedge c$ then $c \leq a \sim b$.

Then $(A, \wedge, \Rightarrow, 1) \in \text{IS}$, where \Rightarrow is defined by $a \Rightarrow b = a \sim (a \wedge b)$. Moreover, we can replace the condition 4) by the inequality

4') $c \wedge ((a \wedge c) \sim (b \wedge c)) \leq a \sim b$.

Therefore, the class of algebras of type $(2, 2, 0)$ which satisfy the conditions 1), 2), 3) and 4) is a variety.

Proof. In order to prove that $(A, \wedge, \Rightarrow, 1) \in \text{IS}$, we only need to prove that for every $a, b, c \in A$, $a \leq b \Rightarrow c$ if and only if $a \wedge b \leq c$. Suppose that $a \leq b \Rightarrow c$, i.e., $a \leq b \sim (b \wedge c)$. Then $a \wedge b \leq b \wedge (b \sim (b \wedge c))$. It follows from 3) that $b \wedge (b \sim (b \wedge c)) = (b \wedge c) \wedge (b \sim (b \wedge c)) \leq c$, so $a \wedge b \leq c$. Conversely, suppose that $a \wedge b \leq c$. Then $a \wedge b = a \wedge (b \wedge c)$. Taking into account 4) we have that $a \leq b \sim (b \wedge c)$, i.e., $a \leq b \Rightarrow c$.

Finally we will prove the equivalence between 4) and 4'). Assume the condition 4). Since $a \wedge (c \wedge ((a \wedge c) \sim (b \wedge c))) = b \wedge (c \wedge ((a \wedge c) \sim (b \wedge c)))$ then $c \wedge ((a \wedge c) \sim (b \wedge c)) \leq a \sim b$, which is the condition 4'). Conversely, assume the condition 4'), and suppose that $a \wedge c = b \wedge c$. It follows from the conditions 2) and 4') that $c = c \wedge ((a \wedge c) \sim (b \wedge c)) \leq a \sim b$, so $c \leq a \sim b$. \square

We write ES for the variety of algebras of type $(2, 2, 0)$ which satisfy the conditions 1), 2), 3) and 4) of Proposition 2.2. The following corollary follows from Proposition 2.2 and Remark 2.1.

Corollary 2.3. 1) *If $(A, \wedge, \rightarrow, 1) \in \text{IS}$ then $(A, \wedge, \leftrightarrow, 1) \in \text{ES}$. Moreover, for every $a, b \in A$ we have that $a \rightarrow b = a \Rightarrow b$, where \Rightarrow is the implication associated to the algebra $(A, \wedge, \leftrightarrow, 1)$.*

2) *If $(A, \wedge, \sim, 1) \in \text{ES}$ then $(A, \wedge, \Rightarrow, 1) \in \text{IS}$. Moreover, for every $a, b \in A$ we have that $a \sim b = a \Leftrightarrow b$.*

3) *The varieties IS and ES are term equivalent.*

3 Hemiimplicative semilattices and symmetric hemiimplicative semilattices

In this section we study the variety of hemiimplicative semilattices and some of its subvarieties. In particular, we present some general examples by defining hemiimplicative structures on any bounded semilattice. In a second part we introduce a new variety, whose algebras will be called for us *symmetric hemiimplicative semilattices*. The original motivation to consider this variety follows from the properties of the algebras $(A, \wedge, \leftrightarrow, 1)$ associated to the algebras $(A, \wedge, \rightarrow, 1) \in \text{hIS}$.

There are several ways of defining a hemiimplicative structure on any bounded semilattice. Some of this ways are described in the examples below. Note that some of this procedures only apply to bounded semilattices with bottom.

Example 3.1. *Let $(A, \wedge, 1)$ be a bounded semilattice (with bottom 0, when necessary). We define binary operations \rightarrow on A that makes the algebra $(A, \wedge, \rightarrow, 1)$ a hemiimplicative semilattice.*

$$a \rightarrow b = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases} \quad (1)$$

$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a \not\leq b \end{cases} \quad (2)$$

In the context of algebras of hIS consider the following two equations:

$$(H4) \quad a \rightarrow (a \wedge b) = a \rightarrow b,$$

$$(H5) \quad (a \wedge b) \rightarrow b = 1.$$

We write hIS_4 for the subvariety of hIS whose algebras satisfy (H4), and hIS_5 for the subvariety of hIS whose algebras satisfy (H5).

Remark 3.2. Let $A \in \text{hIS}_4$ and $a \in A$. Then $a \rightarrow 1 = 1$. It follows from that $a \rightarrow 1 = a \rightarrow (a \wedge 1)$ and $a \rightarrow a = 1$.

Proposition 3.3. *The following inclusions of varieties are proper: $\text{hIS}_4 \subseteq \text{hIS}_5 \subseteq \text{hIS}$.*

Proof. In order to prove that hIS_4 is a subvariety of hIS_5 , let $A \in \text{hIS}_4$ and let $a, b \in A$. Then $(a \wedge b) \rightarrow b = (a \wedge b) \rightarrow ((a \wedge b) \wedge b)$. But $(a \wedge b) \wedge b = a \wedge b$ and $(a \wedge b) \rightarrow (a \wedge b) = 1$. Then $(a \rightarrow b) \rightarrow b = 1$, so $A \in \text{hIS}_5$. In order to show that hIS_4 is a proper subvariety of hIS_5 , consider the boolean lattice B_4 of four elements, where x and y are the atoms, and consider the implication given in (2) of Example 3.1. Then $(a \wedge b) \rightarrow b = 1$ for every a, b . However, $x \rightarrow (x \wedge y) = x \rightarrow 0 = 0$ and $x \rightarrow y = y$, so $x \rightarrow (x \wedge y) \neq x \rightarrow y$. Thus, hIS_4 is a proper subvariety of hIS_5 .

Finally we shall show that hIS_5 is a proper subvariety of hIS. Consider B_4 with the implication given in (1) of Example 3.1. Then $(x \wedge y) \rightarrow y = 0$. Therefore, the equation (H5) is not satisfied. \square

Let $A \in \text{hIS}$ and $a, b \in A$. Notice that if $a \rightarrow b = 1$ then $a \leq b$ because $a = a \wedge 1 = a \wedge (a \rightarrow b) \leq b$. In the following corollary we characterize the hemiimplicative semilattices in which holds the converse property, that is to say, the condition if $a \leq b$ then $a \rightarrow b = 1$.

Corollary 3.4. *Let $A \in \text{hIS}$. The following conditions are equivalent:*

- 1) $A \in \text{hIS}_5$.
- 2) For every $a, b \in A$, $a \leq b$ if and only if $a \rightarrow b = 1$.

Proof. Suppose that $A \in \text{hIS}_5$. Let $a \leq b$. Then $1 = (a \wedge b) \rightarrow b = a \rightarrow b$. Conversely, suppose that for every $a, b \in A$, $a \leq b$ if and only if $a \rightarrow b = 1$. Since $a \wedge b \leq b$ then $(a \wedge b) \rightarrow b = 1$. Therefore, $A \in \text{hIS}_5$. \square

Example 3.5. Let $(A, \wedge, 1)$ be a bounded semilattice (with bottom 0, when necessary). We define binary operations \rightarrow on A that makes the algebra $(A, \wedge, \rightarrow, 1)$ a hemiimplicative semilattice.

$$a \rightarrow b = \begin{cases} 1 & \text{if } a = b \\ b & \text{if } a \neq b \end{cases} \quad (3)$$

$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b \\ a \wedge b & \text{if } a \not\leq b \end{cases} \quad (4)$$

$$a \rightarrow b = \begin{cases} 1 & \text{if } a = b \\ a \wedge b & \text{if } a \neq b \end{cases} \quad (5)$$

$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{if } a \not\leq b \end{cases} \quad (6)$$

In the examples 3.1 and 3.5 we define a binary operation that makes an algebra a hemiimplicative semilattice. In the rest of the paper we shall refer to this operation as the implication of the algebra.

Remark 3.6. The algebras with the implications (4) and (6) of Example 3.5 satisfy (H4). The algebras with the implication (2) of Example 3.1 satisfy (H5), and the algebras with the implication (3) of Example 3.5 also satisfy (H5). For the algebras with the implication (1) of Example 3.1 and by (5) of Example 3.5, if the universe of them is not trivial, then (H5) is not satisfied because $(0 \wedge 1) \rightarrow 1 = 0$. For the algebras with the implication (2) of Example 3.1 and with the implication (3) of Example 3.5 we have that $1 \rightarrow (1 \wedge 0) = 0$ and $1 \rightarrow 1 = 1$, so (H4) is not satisfied by them.

Lemma 3.7. Let $A \in \text{hIS}$ and $a, b \in A$. Then $a \leftrightarrow b = b \leftrightarrow a$, $a \leftrightarrow a = 1$ and $a \wedge (a \leftrightarrow b) \leq b$.

Inspired by Lemma 3.7 we introduce the following variety of algebras.

Definition 3.8. We say that $(A, \sim, \wedge, 1)$ is a *symmetric hemiimplicative semilattice* if $(A, \wedge, \sim, 1) \in \text{hIS}$ and $a \sim b = b \sim a$ for every $a, b \in A$. We write ShIS for the variety of symmetric hemiimplicative semilattices.

Let $A \in \text{ShIS}$. Define the binary operation \Rightarrow by $a \Rightarrow b := a \sim (a \wedge b)$.

Lemma 3.9. Let $A \in \text{ShIS}$ and $a, b \in A$. Then

- 1) $a \Rightarrow a = 1$,
- 2) $a \wedge (a \Rightarrow b) \leq b$,
- 3) $a \Rightarrow (a \wedge b) = a \Rightarrow b$,
- 4) If $b \leq a$ then $a \leftrightarrow b = a \Rightarrow b = a \sim b$.

Proof. Let $a, b \in A$. Then $a \Rightarrow a = a \rightarrow a = 1$, so $a \Rightarrow a = 1$. Besides $a \wedge (a \Rightarrow b) = a \wedge (a \sim (a \wedge b)) \leq a \wedge b \leq b$. By definition of \Rightarrow we have that $a \Rightarrow (a \wedge b) = a \Rightarrow b$.

Finally we shall prove that if $b \leq a$ then $a \Leftrightarrow b = a \Rightarrow b = a \sim b$. In order to show it, let $b \leq a$. We have that

$$\begin{aligned} a \Leftrightarrow b &= (a \Rightarrow b) \wedge (b \Rightarrow a) \\ &= (a \sim (a \wedge b)) \wedge (b \sim (b \wedge a)) \\ &= (a \sim b) \wedge (b \sim b) \\ &= (a \sim b) \wedge 1 \\ &= a \sim b. \end{aligned}$$

Moreover, $a \sim b = a \sim (a \wedge b) = a \Rightarrow b$. Thus we obtain that if $b \leq a$ then $a \Leftrightarrow b = a \sim b = a \Rightarrow b$. \square

In what follows we establish the relation between the varieties hIS and ShIS.

Proposition 3.10. 1) If $(A, \rightarrow, \wedge, 1) \in \text{hIS}$ then $(A, \wedge, \leftrightarrow, 1) \in \text{ShIS}$.

2) If $(A, \rightarrow, \wedge, 1) \in \text{hIS}_4$ then $a \rightarrow b = a \Rightarrow b$ for every $a, b \in A$, where \Rightarrow is the implication associated to the algebra $(A, \wedge, \leftrightarrow, 1)$.

3) If $(A, \sim, \wedge, 1) \in \text{ShIS}$ then $(A, \Rightarrow, \wedge, 1) \in \text{hIS}_4$.

We write ShIS_E for the subvariety of ShIS whose algebras satisfy the following condition:

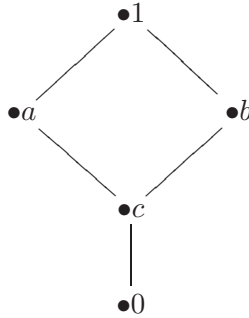
$$(S) \quad a \sim b = (a \sim (a \wedge b)) \wedge (b \sim (a \wedge b)).$$

Equation (S) simply states that in $(A, \sim, \wedge, 1) \in \text{ShIS}_E$, we have that $a \Leftrightarrow b = a \sim b$ for every $a, b \in A$, where \Rightarrow is the implication associated to the algebra $(A, \wedge, \sim, 1)$.

Corollary 3.11. The varieties hIS_4 and ShIS_E are term equivalent.

It is the case that ShIS_E is a proper subvariety of ShIS, as the following example shows.

Example 3.12. Let A be the bounded lattice of the following figure:



Define on A the following binary operation:

\sim	0	a	b	c	1
0	1	0	0	0	0
a	0	1	0	c	a
b	0	0	1	c	b
c	0	c	c	1	c
1	0	a	b	c	1

Straightforward computations show that $(A, \wedge, \sim, 1) \in \text{ShIS}$. However, $a \sim b = 0$, $a \sim (a \wedge b) = c$ and $b \sim (a \wedge b) = c$. Thus, we obtain that $a \sim b \neq (a \sim (a \wedge b)) \wedge (b \sim (a \wedge b))$. Therefore, $(A, \wedge, \sim, 1) \notin \text{ShIS}_E$.

Remark 3.13. For $i = 1, \dots, 6$, let K_i be the class of the algebras in hIS where the implication is given by (i) of examples 3.1 and 3.5. It can be proved that these classes are not closed by products; hence, they are not quasivarieties. It would be interesting to have an answer for the following general question: which is a set of equations (quasiequations) for the variety (quasivariety) generated by K_i ?

4 Congruences

In this section we study the congruences for some subclasses of hIS. More precisely, in Subsection 4.1 we study the lattice of congruences for the algebras given in examples 3.1 and 3.5. In Subsection 4.2 we characterize the principal congruences in hIS.

In what follows we fix notation and we give some definitions. We also give some known results about congruences on hemiimplicative semilattices [14]. Let $A \in \text{hIS}$, $a, b \in A$ and θ a congruence of A . We write a/θ to indicate the equivalence class of a associated to the congruence θ and $\theta(a, b)$ for the congruence generated by the pair (a, b) . Let $A \in \text{hIS}$ or $A \in \text{ShIS}$. As usual, we say that F is a *filter* if it is a subset of A such satisfies the following conditions: $1 \in F$, if $a, b \in F$ then $a \wedge b \in F$, if $a \in F$ and $a \leq b$ then $b \in F$. We also consider the binary relation

$$\Theta(F) = \{(a, b) \in A \times A : a \wedge f = b \wedge f \text{ for some } f \in F\}.$$

Notice that if A is a semilattice with greatest element (or a bounded semilattice) and F is a filter, then $\Theta(F)$ is the congruence associated to the filter F . For $A \in \text{hIS}$ and $a, b, f \in A$ we define the following element of A : $t(a, b, f) := (a \rightarrow b) \leftrightarrow ((a \wedge f) \rightarrow (b \wedge f))$.

Definition 4.1. Let $A \in \text{hIS}$ and F a filter of A . We say that F is a *congruent filter* if $t(a, b, f) \in F$ for every $a, b \in A$ and $f \in F$.

The following result was proved in [14].

Theorem 4.2. *Let $A \in \text{hIS}$. There exists an isomorphism between the lattice of congruences of A and the lattice of congruent filters of A , which is established via the assignments $\theta \mapsto 1/\theta$ and $F \mapsto \Theta(F)$.*

Notice that if $(A, \wedge, \sim, 1)$ is a symmetric hemiimplicative semilattice and $a, b, f \in F$ then $t(a, b, f) = (a \sim b) \sim ((a \wedge f) \sim (b \wedge f))$.

Corollary 4.3. *Let $A \in \text{ShIS}$. There exists an isomorphism between the lattice of congruences of A and the lattice of filters F of A which satisfy $(a \sim b) \sim ((a \wedge f) \sim (b \wedge f)) \in F$ for every $a, b \in A$ and $f \in F$. The isomorphism is established via the assignments $\theta \mapsto 1/\theta$ and $F \mapsto \Theta(F)$.*

4.1 Congruences for some algebras of hIS

In this subsection we characterize the congruences of examples 3.1 and 3.5.

The following elemental remark will be used later.

Remark 4.4. Let $A \in \text{hIS}$ and F a congruent filter of A . Then $F = \{1\}$ if and only if $x/\Theta(F) = \{x\}$ for every x .

Congruent filters associated to the implications given by (1) or (6)

Let F be a filter of $(A, \wedge, \rightarrow, 1) \in \text{hIS}$, where \rightarrow is the implication given by (1) or (6).

Proposition 4.5. *F is congruent if and only if $F = \{1\}$ or $F = A$.*

Proof. If the algebra A is trivial the proof is immediate, so we can assume that A is an algebra not trivial. Suppose that F is congruent, and that $F \neq \{1\}$. By Remark 4.4 we have that there exist $x, y \in A$ such that $x \neq y$ and $x/\Theta(F) = y/\Theta(F)$. Thus, there is f in F such that $x \wedge f = y \wedge f$. Suppose that $x \not\leq y$. Taking into account that F is congruent we have that $(x \rightarrow y) \rightarrow 1 \in F$. But $x \not\leq y$, so $x \rightarrow y = 0$. Hence, $(x \rightarrow y) \rightarrow 1 = 0 \rightarrow 1$. However, $0 \rightarrow 1 = 0$ because $0 \not\leq 1$. Then we have that $0 \in F$. Analogously we can show that if $y \not\leq x$ then $0 \in F$ by considering $y \rightarrow x$ in place of $x \rightarrow y$. Therefore, $F = A$. \square

Congruent filters associated to the implication given by (2)

Let F be a filter of $(A, \wedge, \rightarrow, 1)$, where \rightarrow is the implication given by (2).

Proposition 4.6. *F is congruent if and only if it satisfies the following conditions for every $x, y \in A$ and $f \in F$:*

- 1) *If $x \not\leq y$ and $x \wedge f \leq y \wedge f$ then $y \in F$.*

2) If $x \not\leq y$, $x \wedge f \not\leq y \wedge f$ and $y \not\leq f$ then $y \in F$.

3) If $x \leq y$ and $x \wedge f \not\leq y \wedge f$ then $y \in F$.

Proof. Let $x, y \in A$ and $f \in F$. Let $x \not\leq y$ and $x \wedge f \leq y \wedge f$. As $t(x, y, f) = y \in F$ we have the condition 1). Suppose that $x \not\leq y$, $x \wedge f \leq y \wedge f$ and $y \not\leq f$. Taking into account that $t(x, y, f) = y \wedge f \in F$ we obtain the condition 2). Finally suppose that $x \leq y$ and $x \wedge f \leq y \wedge f$. Since $t(x, y, f) = y \wedge f \in F$ then we obtain 3).

Conversely, suppose that we have the conditions 1), 2) and 3). Let $x, y \in A$ and $f \in F$. Suppose that $x \leq y$. If $x \wedge f \leq y \wedge f$ then $t(x, y, f) = 1 \in F$. If $x \wedge f \not\leq y \wedge f$ then $t(x, y, f) = y \wedge f$. By 3) we have that $y \in F$, so $t(x, y, f) \in F$. Suppose now that $x \not\leq y$. If $x \wedge f \not\leq y \wedge f$ then $t(x, y, f) = y \in F$ by 1). If $x \wedge f \leq y \wedge f$ then $t(x, y, f) = y \leftrightarrow (y \wedge f)$. We have two possibilities: $y \leq f$ or $y \not\leq f$. If $y \leq f$ then $t(x, y, f) = 1 \in F$. If $y \not\leq f$ then $t(x, y, f) = y \wedge f$. By 2) we have that $y \in F$. Hence, $t(x, y, f) \in F$. Therefore, F is congruent. \square

Congruent filters associated to the implication given by (3)

Let F be a filter of $(A, \wedge, \rightarrow, 1)$, where \rightarrow is the implication given by (3).

Proposition 4.7. *F is congruent if and only if it satisfies the following conditions for every $x, y \in A$ and $f \in F$:*

1) If $x \neq y$ and $x \wedge f = y \wedge f$ then $y \in F$.

2) If $x \neq y$, $x \wedge f \neq y \wedge f$ and $y \not\leq f$ then $y \in F$.

Proof. Assume that F is congruent, and let $x, y \in A$ such that $x \neq y$. Let $f \in F$. Then $t(x, y, f) = y \leftrightarrow ((x \wedge f) \rightarrow (y \wedge f))$. Moreover, $t(x, y, f) \in F$. If $x \wedge f = y \wedge f$ then $t(x, y, f) = y \in F$. Suppose now that $x \wedge f \neq y \wedge f$ and $y \not\leq f$ (i.e., $y \neq y \wedge f$). Thus, $t(x, y, f) = y \leftrightarrow (y \wedge f) = y \wedge f \in F$, so $y \in F$.

Conversely, suppose it holds the conditions 1) and 2). Consider $x, y \in A$ and $f \in F$. If $x = y$ then $t(x, y, f) = 1 \in F$. If $x \neq y$ then $t(x, y, f) = y \leftrightarrow ((x \wedge f) \rightarrow (y \wedge f))$. If $x \wedge f = y \wedge f$ then $t(x, y, f) = y$, which belongs to F by condition 1). Suppose that $x \wedge f \neq y \wedge f$. Then $t(x, y, f) = y \leftrightarrow (y \wedge f)$. If $y \leq f$ then $t(x, y, f) = 1 \in F$. If $y \not\leq f$ then $t(x, y, f) = y \wedge f$, which also belongs to F by condition 2) (because $f, y \in F$). \square

Congruent filters associated to the implication given by (4)

Let F be a filter of $(A, \wedge, \rightarrow, 1)$, where \rightarrow is the implication given by (4).

Proposition 4.8. *F is congruent if and only if it satisfies the following conditions for every $x, y \in A$ and $f \in F$:*

1) If $x \not\leq y$ and $x \wedge f \leq y \wedge f$ then $x \wedge y \in F$.

2) If $x \not\leq y$, $x \wedge f \not\leq y \wedge f$ and $x \wedge y \not\leq f$ then $x \wedge y \in F$.

Proof. Suppose that F is congruent. Let $x \not\leq y$ and $x \wedge f \leq y \wedge f$. Then $t(x, y, f) = x \wedge y \in F$, so we have the condition 1). Suppose that $x \not\leq y$, $x \wedge f \not\leq y \wedge f$ and $x \wedge y \not\leq f$. Then $t(x, y, f) = x \wedge y \wedge f \in F$, so $x \wedge y \in F$, which is the condition 2).

Conversely, suppose that it holds the conditions 1) and 2). Let $x, y \in A$ and $f \in F$. If $x \leq y$ then $t(x, y, f) = 1 \in F$. Suppose that $x \not\leq y$. If $x \wedge f \leq y \wedge f$ then $t(x, y, f) = x \wedge y$, which belongs to F by the condition 1). Suppose that $x \not\leq y$ and $x \wedge f \not\leq y \wedge f$. Then $t(x, y, f) = (x \wedge y) \leftrightarrow (x \wedge y \wedge f)$. If $x \wedge y \leq f$ then $t(x, y, f) = 1 \in F$. If $x \wedge y \not\leq f$ then $t(x, y, f) = x \wedge y \wedge f$. But by condition 2) we have that $x \wedge y \in F$. Therefore, $t(x, y, f) \in F$. \square

Congruent filters associated to the implication given by (5)

Let F be a filter of $(A, \wedge, \rightarrow, 1)$, where \rightarrow is the implication given by (5).

Proposition 4.9. *F is congruent if and only if it satisfies the following conditions for every $x, y \in A$ and $f \in F$:*

1) If $x \neq y$ and $x \wedge f = y \wedge f$ then $x \wedge y \in F$.

2) If $x \neq y$, $x \wedge f \neq y \wedge f$ and $x \wedge y \not\leq f$ then $x \wedge y \in F$.

Proof. Suppose that F is congruent. In order to prove 1) and 2) consider $x \neq y$. Suppose that $x \wedge f = y \wedge f$. Then we have that $t(x, y, f) = (x \wedge y) \leftrightarrow 1$. If $1 \leftrightarrow (x \wedge y) = 1$ then $1 = x \wedge y$, i.e., $x = y = 1$, which is an absurd. Then $t(x, y, f) = x \wedge y \in F$. Hence we have the condition 1). Suppose now that $x \wedge f \neq y \wedge f$ and $x \wedge y \not\leq f$. Hence, $t(x, y, f) = (x \wedge y) \leftrightarrow (x \wedge y \wedge f)$. But $x \wedge y \not\leq f$, so $t(x, y, f) = x \wedge y \wedge f \in F$. Thus $x \wedge y \in F$, which is the condition 2).

Conversely, suppose that F satisfies the conditions 1) and 2). Let $x, y \in F$ and $f \in F$. If $x = y$ then $t(x, y, f) = 1 \in F$. Suppose that $x \neq y$, so $t(x, y, f) = (x \wedge y) \leftrightarrow ((x \wedge f) \rightarrow (y \wedge f))$. If $x \wedge f = y \wedge f$ then $t(x, y, f) = x \wedge y$, which belongs to F by 1). Suppose that $x \wedge f \neq y \wedge f$, so $t(x, y, f) = (x \wedge f) \leftrightarrow (x \wedge y \wedge f)$. If $x \wedge y \leq f$ then $t(x, y, f) = 1 \in F$. If $x \wedge y \not\leq f$ then by 2) we have that $x \wedge y \in F$. But $t(x, y, f) = x \wedge y \wedge f$. Thus, $t(x, y, f) \in F$. \square

Totally ordered posets

Let $A \in \text{hIS}$ and $x, y, f \in A$. If $f \leq x \wedge y$ then $t(x, y, f) = (x \rightarrow y) \leftrightarrow 1$, if $y \leq f \leq x$ then $t(x, y, f) = (x \rightarrow y) \leftrightarrow (f \rightarrow y)$, if $x \leq f \leq y$ then $t(x, y, f) = (x \rightarrow y) \leftrightarrow (x \rightarrow f)$ and if $x \leq f$ and $y \leq f$ then $t(x, y, f) = 1$. Hence, we obtain the following result.

Proposition 4.10. *Let $A \in \text{hIS}$ such that its underlying poset is a chain and let F be a filter of A . Then F is congruent if and only if for every $x, y \in A$ and $f \in F$ it holds the following conditions:*

- (a) *If $f \leq x \wedge y$ then $(x \rightarrow y) \leftrightarrow 1 \in F$.*
- (b) *If $y \leq f \leq x$ then $(x \rightarrow y) \leftrightarrow (f \rightarrow y) \in F$.*
- (c) *If $x \leq f \leq y$ then $(x \rightarrow y) \leftrightarrow (x \rightarrow f) \in F$.*

Corollary 4.11. *Consider the hemiimplicative semilattices, whose underlying order is total, with the implications (2), (3), (4) and (5) (see examples 3.1 and 3.5). Every filter in these algebras is congruent.*

Proof. Let $x, y \in A$ and $f \in F$. If $f \leq x \wedge y$. Then $(x \rightarrow y) \leftrightarrow 1 \in \{x \wedge y, y, 1\} \subseteq F$. If $y \leq f \leq x$ then $(x \rightarrow y) \leftrightarrow (f \rightarrow y) \in \{f, 1\} \subseteq F$. If $x \leq f \leq y$ then $(x \rightarrow y) \leftrightarrow (x \rightarrow f) \in \{f, 1\} \subseteq F$. Therefore, it follows from Proposition 4.10 that F is congruent. \square

Notice, however, that it follows from Proposition 4.5 that there are examples of hemiimplicative semilattices where the order is total with the property that not every filter is congruent.

On the other hand, it is not the case that every filter in a non totally ordered algebra of the classes considered in previous corollary is congruent. Consider, for example, the boolean lattice of four elements, where x and y are the atoms, and let $F = \{x, 1\}$. We write B for the universe of this algebra. We have that F is a filter of $(B, \wedge, \rightarrow_i, 1)$, where \rightarrow_i is the implication given in (i) of the above mentioned examples, for $i = 1, \dots, 6$. We write t_i for the ternary term t over the algebra $(B, \wedge, \rightarrow_i, 1)$. Since $t_1(x, y, x)$, $t_2(y, x, x)$, $t_3(0, y, x)$, $t_4(0, y, x)$, $t_5(y, x, x)$ and $t_6(x, y, x)$ are the bottom then F is not a congruent filter.

4.2 Principal congruences for algebras of hIS

Let $A \in \text{hIS}$ and $a, b \in A$. We write $F^c(a)$ for the congruent filter generated by $\{a\}$. In [14] it was proved that if $\theta(a, b)$ is the congruence generated by (a, b) , then $(x, y) \in \theta(a, b)$ if and only if $x \leftrightarrow y \in F^c(a \leftrightarrow b)$. We will give some definitions in order to make possible an explicit description of $F^c(a)$.

For $X \subseteq A$ we define $t(x, y, X) = \{t(x, y, z) : z \in X\}$. Then we define $t^+(x, y, X)$ as the elements $z \in A$ such that $z \geq t(x, y, w_1) \wedge \dots \wedge t(x, y, w_k) \wedge w_{k+1} \wedge \dots \wedge w_{k+t}$ for some $w_i \in X$. In a next step we define

$$t(X) = \bigcup_{x, y \in A} t^+(x, y, X).$$

We also define $T_0(X) = X$, $T_{n+1}(X) = t(T_n(X))$ and $T(X) = \bigcup_{n \in \mathbb{N}} T_n(X)$, where \mathbb{N} is the set of natural numbers. It is immediate that $a \in T(\{a\})$.

Proposition 4.12. *Let $A \in \text{hIS}$ and $a \in A$. Then $F^c(a) = T(\{a\})$.*

Proof. Straightforward computations based on induction show that

$$T_n(\{a\}) \subseteq T_{n+1}(\{a\}) \quad (7)$$

for every n . We use this property throughout this proof.

We have that $1 \in T(\{a\})$. It follows from the construction that $T(\{a\})$ is an upset. In order to prove that this set is closed by \wedge , let $z, z' \in T(\{a\})$. Then there are n and m such that $z \in T_n(\{a\})$ and $z' \in T_m(\{a\})$. By (7) we have that $z, z' \in T_p(\{a\})$, where p is the maximum between n and m . Straightforward computations prove that $z \wedge z' \in T_p(\{a\})$, so $z \wedge z' \in T(\{a\})$. Hence, $T(\{a\})$ is a filter.

In order to show that $T(\{a\})$ is congruent, let $z \in T(\{a\})$ and $x, y \in A$. Then, there is n such that $z \in T_n(\{a\})$. Taking into account that $t(x, y, z) \geq t(x, y, z) \wedge z$, we have that $t(x, y, z) \in t(T_n(\{a\}))$, i.e., $t(x, y, z) \in T_{n+1}(\{a\}) \subseteq T(\{a\})$. Thus, $T(\{a\})$ is congruent.

Finally we show that $F^c(a) = T(\{a\})$. Let F be a congruent filter such that $a \in F$. We need to prove that $T(\{a\}) \subseteq F$, i.e., that $T_n(\{a\}) \subseteq T(\{a\})$ for every n . If $n = 0$ is immediate. Suppose that $T_n(\{a\}) \subseteq T(\{a\})$ for some n . We shall prove that $T_{n+1}(\{a\}) \subseteq T(\{a\})$. Let $z \in T_{n+1}(\{a\})$. Then there are $x, y \in A$ and $w_1, \dots, w_{k+t} \in T_n(\{a\})$ such that

$$z \geq t(x, y, w_1) \wedge \dots \wedge t(x, y, w_k) \wedge w_{k+1} \wedge \dots \wedge w_{k+t}.$$

But $T_n(\{a\}) \subseteq F$, and F is a congruent filter. Thus,

$$t(x, y, w_1) \wedge \dots \wedge t(x, y, w_k) \wedge w_{k+1} \wedge \dots \wedge w_{k+t} \in F.$$

Hence, $z \in F$. Therefore, $T_{n+1}(\{a\}) \subseteq F$, which was our aim. \square

Corollary 4.13. *Let $A \in \text{hIS}$ and $a, b \in A$. Then $(x, y) \in \theta(a, b)$ if and only if $x \leftrightarrow y \in T(\{a \leftrightarrow b\})$.*

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References

- [1] Celani S.A. and Jansana R., *Bounded distributive lattices with strict implication*. Mathematical Logic Quarterly 51, No. 3, 219–246 (2005).
- [2] Celani S.A., Castiglioni J.L. and San Martín H.J., *Kleene algebras with implication*. Algebra Universalis (accepted in 2016).

- [3] Ciungu L.C., *On pseudo equality algebras*. Archive for Mathematical Logic 53, No. 5, 561–570 (2014).
- [4] Curry H.B., *Foundations of mathematical logic*. McGraw-Hill, New York (1963).
- [5] Dvurecenskij A. and Zahir, *Pseudo equality algebras: revision*. Soft Computing 20, No. 6, 2091–2101 (2016).
- [6] Idziak P.M., *Lattice operations in BCK-algebras*. Math Japonica 29, 839–846 (1984).
- [7] Jeney S., *Equality algebras*. Studia Logica 100, 1201–1209 (2012).
- [8] Jeney S., *Pseudo equality algebras*. Archive for Mathematical Logic 52, No. 3, 469–481 (2013).
- [9] Kühr J., *Pseudo BCK-algebras and related structures*. Univerzita Palackého Olomouci (2007).
- [10] Nemitz W., *Implicative semi-lattices*. Trans. Amer. Math. Soc. 117, 128–142 (1965).
- [11] Sankappanavar H.P., *Semi-Heyting algebras: an abstraction from Heyting algebras*. Proceedings of the 9th Congreso “Dr. Antonio A. R.”, 33–66, Actas Congr. “Dr. Antonio A. R. Monteiro”, Univ. Nac. del Sur, Bahía Blanca, Argentina (2008).
- [12] Sankappanavar H.P., *Expansions of Semi-Heyting Algebras I: Discriminator Varieties*, Studia Logica 98, No. 1–2, 27–81 (2011).
- [13] San Martín, H.J., *Compatible operations on commutative weak residuated lattices*. Algebra Universalis 73, No. 2, 143–155 (2015).
- [14] San Martín H.J., *Congruences in weak implicative semi-lattices*. Soft Computing DOI 10.1007/s00500-016-2188-9 (accepted in 2016).

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